ENGSCI 332
Control Systems

Lecture 4
Feedback System Analysis

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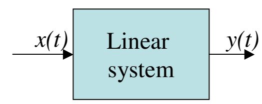
Outline

- Frequency analysis and concepts
  - Linear time-invariant systems
  - Sinusoidal / exponential basis functions
  - Gain and phase transformations
- Impulse response
- Convolution
- Bode plot approximations

Linear systems

- Consider a system that causes only linear changes to an input signal e.g. $y(t) = Ax(t-b)$ where A and b are constants
- **Principle of superposition:** if $y_1(t) = Ax_1(t)$ and $y_2(t) = Ax_2(t)$
- Then
  $y_1(t) + y_2(t) = Ax_1(t) + Ax_2(t)$
  $= A[x_1(t) + x_2(t)]$
  Time Invariance



- If $y(t) = F(x(t))$, then $F(x(t-t_1)) = y(t-t_1)$

- Therefore, analysis of linear systems can treat each component separately
  - i.e. recall Laplace Transform – solution to differential equations is sum of complex exponentials.
  - Can also represent signal by Laplace transform
  - So can consider each of the constituent parts separately
    - How does system change each constituent?
    - Total effect is sum of individual effects.

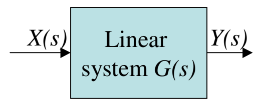
Laplace revisited

- Laplace transform
  - Derived from solution to differential equation as sum of exponential basis functions
  - i.e. Laplace transfer function represents system response as sum of (complex) exponentials
  - If we also represent a signal by sum of (complex) exponential basis functions, then, by principle of superposition, can simply obtain the effect of the system by multiplying each of the constituent components by the system's response to that exponential basis function.
  - System Transfer Function – use algebra to represent the effect of a linear system on input signal

System Transfer Function

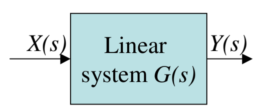
- Signal *x(t)* characterised by its transform *X(s)* – complex amplitude of exponentials.
- Each exponential component is modified by linear system, so output signal transform *Y(s) = G(s).X(s)*
- G(s) is the *transfer function*, which is complex. In effect, the *magnitude* of G(s) modifies the amplitude of the component, and the *phase* of G(s) (recall the phasor representation of the complex exponential) modifies the phase of the corresponding component.

  i.e. if $x(t) = \sin(wt)$, then $y(t) = A \sin(wt + \phi)$
  where $A = |G(jw)|$ and $\phi = \arg\{G(jw)\}$



Convolution

- What is relationship between time and frequency-domain representation of transfer function?
- Take Fourier Transform
- Substitute in FT of H(*f*) and rearrange



$$Y(s) = X(s)G(s)$$

$$y(t) = \int_{-\infty}^{\infty} X(f)H(f)e^{j2\pi ft}df$$

$$y(t) = \int_{-\infty}^{\infty} X(f)\left[\int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f\tau}d\tau\right]e^{j2\pi ft}df$$
$$= \int_{-\infty}^{\infty} h(\tau)\left[\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}e^{-j2\pi f\tau}df\right]d\tau$$
$$= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

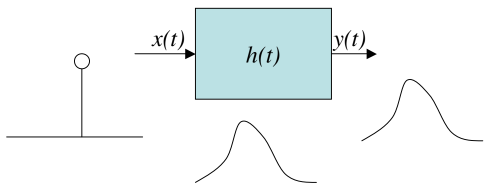
$$y(t) = h(t) * x(t)$$

- Multiplication in frequency domain is convolution in time domain

Convolution – Impulse response

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

- Convolution integral says that the output of the system at time *t* is related to the input by a weighted integral over the duration of the function *h*(*t*).
  i.e. if *x*(*t*) was a single impulse occurring at *t*=0, and zero elsewhere, then *y*(*t*) = *h*(*t*)
- *h(t)* is termed the *impulse response* or *point spread function* (2-D) of the system
- Because of linearity property, impulse response characterises effect of system on the signal – i.e. *x*(*t*) can be composed of a series of impulses, and so *y*(*t*) is series of weighted and summed impulse responses.



Impulse response

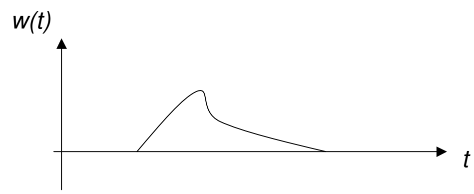
- Define unit input functions:
  – Unit step: $u(t) = \begin{cases} 0, & t \le 0 \\ 1, & t > 0 \end{cases}$
  – Unit impulse $\delta(t) = \lim_{\Delta t \to 0} \left[\frac{u(t) - u(t-\Delta t)}{\Delta t} \right]$ $\int_{-\infty}^{t} \delta(\tau - t_0)d\tau = u(t - t_0)$
- Impulse function has very important *screening* property

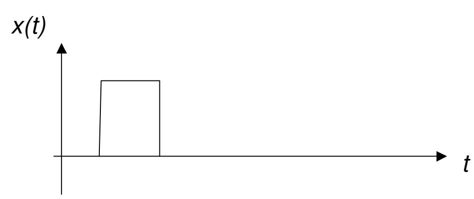
$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$$

- Impulse response:
  – System output when all initial conditions set to zero : $\frac{d^n}{dt^n} y(0) = 0, \quad n = 0...$
  and input is unit impulse $x(t) = \delta(t)$

$$y_\delta(t) = \int_0^t w(t-\tau)\delta(\tau)d\tau \quad = \int_{-\infty}^{\infty} w(t-\tau)\delta(\tau)d\tau \quad = w(t)$$

- Implication is that, since any function can be expressed in terms of weighted impulse function, we only need to know response of system to an impulse and we can calculate its response to any input



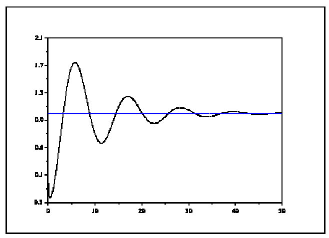


$$y(t)=\int_{-\infty}^{\infty} w(\tau)x(t-\tau)d\tau$$

Transient response

- Need to be able to predict the response of a system to some disturbance on the input
- i.e. step response – input suddenly changes
  - System output when input is equal to unit step (eg what happens when input switched on)

$$y_u(t) = \int_0^t w(t-\tau)u(\tau)d\tau$$



2nd order system

- For purpose of understanding, write 2nd order differential equation:

$$\frac{d^2y}{dt^2} + 2\varsigma\omega_n \frac{dy}{dt} + \omega_n^2 y = \omega_n^2 x$$

where ζ is *damping ratio* and ω_n is *natural frequency* of the system.

- – Characteristic equation then

$$D^2 + 2\varsigma\omega_n D + \omega_n^2 = 0$$

- – with solutions

$$D_1 = -\varsigma\omega_n + j\omega_n\sqrt{1-\varsigma^2} \qquad D_2 = -\varsigma\omega_n - j\omega_n\sqrt{1-\varsigma^2}$$

- – Define *damping coefficient* $\alpha = \zeta\omega_n$

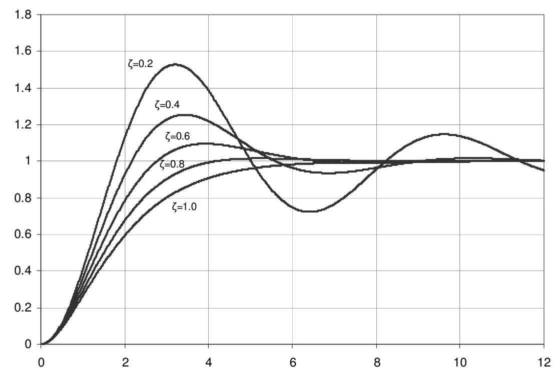
and *damped natural frequency* $\omega_d = \omega_n\sqrt{1-\varsigma^2}$

- – Weighting function then $w(t) = \frac{1}{\omega_d} e^{-\alpha t} \sin\omega_d t$

Step response
- 2nd order system

$$y_u(t) = \int_0^t w(t-\tau)\omega_n^2 d\tau \qquad = 1 - \frac{\omega_n e^{-\alpha t}}{\omega_d}\sin(\omega_d t + \theta), \quad \theta = \tan^{-1}(\omega_d / \alpha)$$

- Damping parameter ζ determines degree of "overshoot"



System transfer

- In general, write transfer function as polynomial
  - Poles and zeros
  - E.g. second order system

$$\frac{d^2y}{dt^2} + a_2\frac{dy}{dt} + a_3 y = b_1 x + b_2\frac{dx}{dt}$$

$$\frac{Y(s)}{X(s)} = \frac{b_1 + b_2 s}{s^2 + a_2 s + a_3}$$
$$= \frac{K(s-\alpha)}{(s-p_1)(s-p_2)}$$

Frequency analysis – gain

- Bode plots – transfer function with respect to frequency
  - Draw a plot of the system response and ensure gain is reasonable at all frequencies
  - E.g. consider 2nd order system $G(s) = \frac{5s+50}{s^2+99.8s-20} = \frac{5(s+10)}{(s-0.2)(s+100)}$
  - Re-arrange $G(s) = \frac{-2.5\left(1+\frac{1}{10}s\right)}{\left(1-\frac{1}{0.2}s\right)\left(1+\frac{1}{100}s\right)}$
  - Then compute log gain (in decibel = $20\log_{10}|G(s)|$

$$20\log|G(s)| = 20\log|-2.5| + 20\log|1+\tfrac{1}{10}s|$$
$$-20\log|1-\tfrac{1}{0.2}s| - 20\log|1+\tfrac{1}{100}s|$$

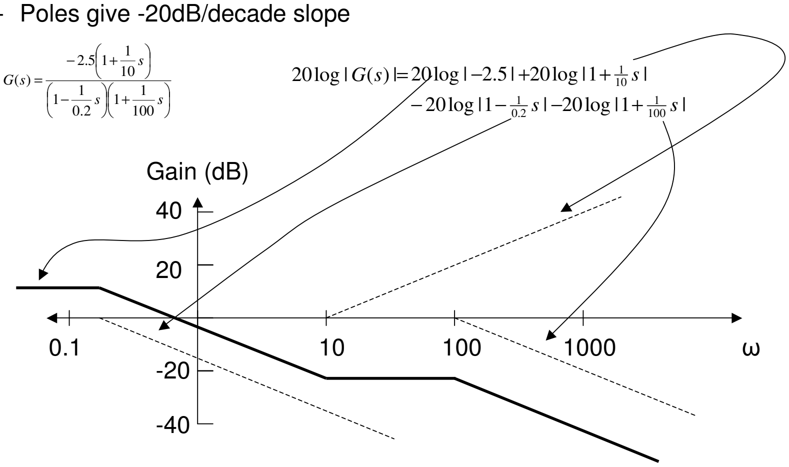
  - Evaluate along frequency axis $j\omega$, noting approximations for $20\log|1+j\omega/\omega_c|$
    - For $\omega < \omega_c$, $20\log|1+j\omega/\omega_c| \approx 20\log(1) = 0$
    - For $\omega > \omega_c$, $20\log|1+j\omega/\omega_c| \approx 20\log \omega/\omega_c$ ie slope 20dB per decade

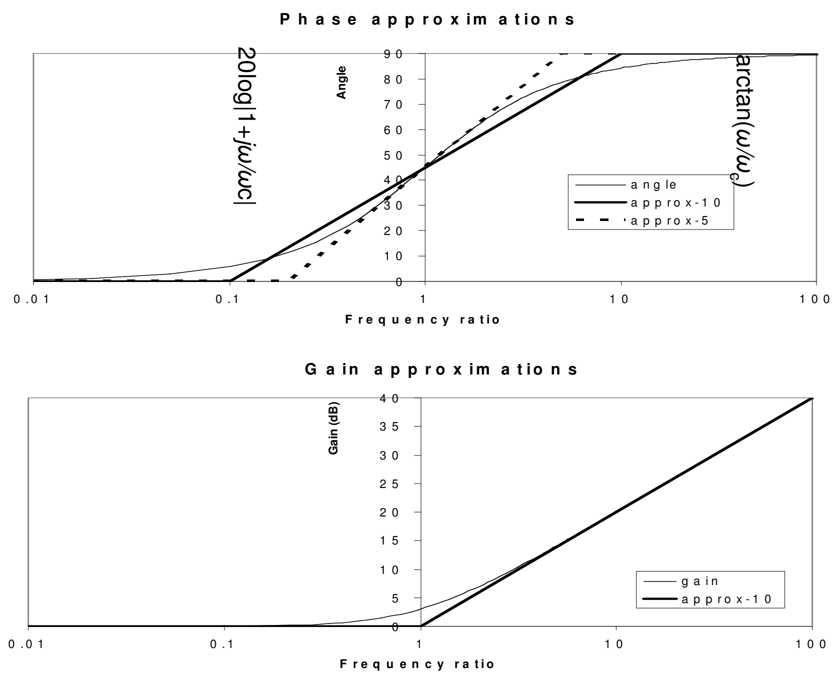
Bode plot

- Approximation: At each "cutoff frequency", change slope accordingly
  - Zeros give +20dB/decade slope
  - Poles give -20dB/decade slope

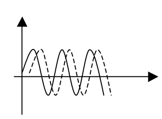
$$G(s) = \frac{-2.5\left(1+\frac{1}{10}s\right)}{\left(1-\frac{1}{0.2}s\right)\left(1+\frac{1}{100}s\right)}$$

$$20\log|G(s)| = 20\log|-2.5| + 20\log|1+\tfrac{1}{10}s|$$
$$-20\log|1-\tfrac{1}{0.2}s| - 20\log|1+\tfrac{1}{100}s|$$





Bode plot - phase



- Phase is the angle of $G(j\omega)$
  - i.e. input $x(t) = \sin(\omega t)$ and output $y(t) = \sin(\omega t+\theta)$ phase difference is θ
- Approximations
  - For $\omega \ll \omega_c$, $\arg(1+j\omega/\omega_c) \approx 0$
  - For $\omega \gg \omega_c$, $\arg(1+j\omega/\omega_c) \approx 90^{\circ}$
  - For $\omega \gg \omega_c$, $\arg(1-j\omega/\omega_c) \approx -90^{\circ}$
  - Transition $\approx \omega/5 < \omega < 5\omega$

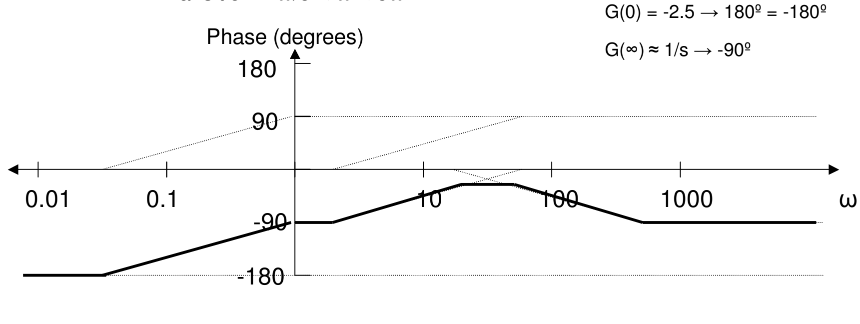
$$G(s) = \frac{-2.5\left(1+\frac{1}{10}s\right)}{\left(1-\frac{1}{0.2}s\right)\left(1+\frac{1}{100}s\right)}$$

(add)

(subtract)

$G(0) = -2.5 \rightarrow 180^{\circ} = -180^{\circ}$

$G(\infty) \approx 1/s \rightarrow -90^{\circ}$



Summary

- Goodwin, Graebe, Salgado: Chapter 5 (analysis),