ENGSCI 332
Control Systems

Lecture 2

System modelling

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Outline

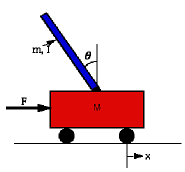
- Modelling systems
  - Mathematical representations
  - Block diagrams
  - Signal “flow”
- Differential equations – time domain
- Laplace transform representation – frequency domain

Modelling systems

- Want to understand behaviour of the system
  - Predict what happens when conditions change
  - Prove that controlled system will do what we want
  - Optimise controller design depending on requirements
- Mathematical representation of the system
  - Can manipulate with mathematical operations
  - Can calculate (ie simulate) what result *should* be
  - Can use mathematics to prove whether it will work
- Obviously, any mathematical calculation or proof is only as good as the model assumptions

System description

- System is generally some physical mechanism
  - E.g. Motor, arm, car, vat, electricity grid



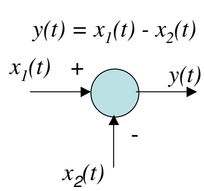
- Individual parts of the system have physical properties
  - E.g. Mass, dimension, temperature, voltage
- Individual parts interact – information/signal flows
  - E.g. Forces, currents, flows, motion

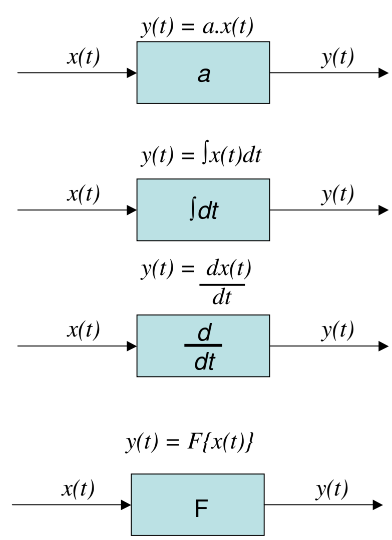
Modelling basics – Mechanics

- Control is generally about making something (physical) happen. E.g.
  - Apply force to move object
  - Put energy into plant to change temperature
- Controller (and plant) can only operate according to fundamental mechanical and thermodynamic laws – e.g.
  - Conservation of momentum / inertia
  - Force -> acceleration
  - Conservation of energy
- Express physical behaviour as mathematical relationships – e.g.
  - Speed of machine is a result of applied forces (less friction) effecting a (rate of) change in momentum / inertia

Block diagrams

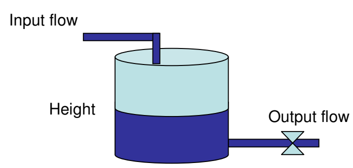
- Represent system components by blocks
- Interactions between blocks comprise *signal flow*
  - Information about system's behaviour
  - System state
- Blocks then transform input signal(s) to output signal

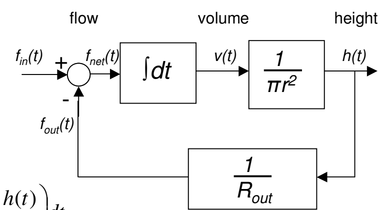




Modelling – block diagrams

- Break down large system into small connected parts
- Represent each part by function between input and output
- Mathematically combine
- For instance, model the level of water in a tank:



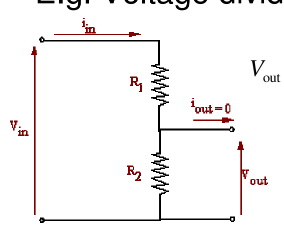


$$h(t) = \frac{1}{\pi r^2}\int\left(f_{in}(t) - \frac{h(t)}{R_{out}}\right)dt$$

$$\frac{dv(t)}{dt} = f_{in}(t) - \frac{v(t)}{\pi r^2 R_{out}}$$

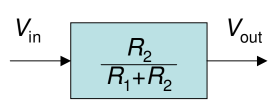
Block diagrams

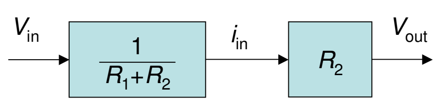
- May model system with more or less detail
- E.g. Voltage divider



The voltage divider circuit

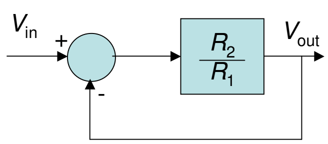
$$V_{out} = V_{in}\frac{R_2}{R_1 + R_2}$$





Rearrange:

$$V_{out}(R_1 + R_2) = V_{in}R_2$$
$$R_1 V_{out} = R_2(V_{in} - V_{out})$$



Time dependence

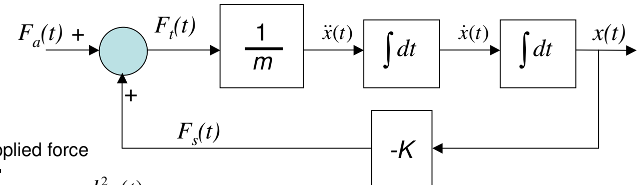
- Components that store energy exhibit *memory*
  - i.e. output depends on past input history
  - E.g. spring, capacitor, inertia, momentum, heat
- Model with conservation of energy laws
  - Differential / integral equations between inputs and outputs

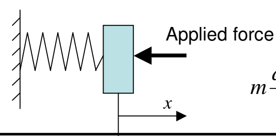
Example

Spring law: $F(t) = -Kx(t)$

Newton's law:

$$F(t) = m\frac{d^2x}{dt^2}$$





$$m\frac{d^2x(t)}{dt^2} = F_a(t) - Kx(t)$$

System memory

- Can represent memory in system by differential equations
  - i.e. rate of change of variable is related to variable (in past)

  e.g. spring: $$m\frac{d^2x}{dt^2} + Kx = F_a$$

  where $F_a=F_a(t)$ is the input signal, and $x=x(t)$ is the output signal
- Solving to find output may not be trivial
  - Requires knowledge of initial conditions - ie $x(0)$, $x'(0)$, etc
  - Also need to know form of input signal

Modelling basics – differential equations

- Generally, control is about making something change (or keeping something constant in spite of changing parameters)
- Differential equations describe how things change in response to other variables. E.g.
  - Velocity is rate of change of position with respect to time
  - Volume (of fluid) is integral over time of fluid flow
  - Chemical reaction may proceed according to differential equation with respect to time and concentration gradients

Differential equations - Characteristic equations

- Consider a differential equation of the form $\sum_{i=0}^{n} a_i \frac{d^i y}{dt^i} = x$
- Can define differential operator $D \equiv \frac{d}{dt}$ and rewrite as:

$$a_0 y + a_1 D^1 y + a_2 D^2 y + \ldots + a_n D^n y = x$$

- Define the *characteristic equation* as polynomial in *D:*

$$a_0 + a_1 D^1 + a_2 D^2 + \ldots + a_n D^n = 0$$

- The characteristic equation has *n* solutions in *D*
  E.g. $\frac{d^2 y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$

  Characteristic equation is $D^2 + 3D + 2 = 0$

  Solutions D_1= -1 and D_2= -2

Differential equations - Solutions

- Recall that $\frac{d}{dt}e^{at} = ae^{at}$
- In general, can form a solution set for differential equation from the characteristic roots: ie

  $y_1=e^{D_1 t}$, $y_2=e^{D_2 t}$, …
- E.g. $\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=0$

  has characteristic equation $D^2+3D+2=0$

  so fundamental solution set is $y_1=e^{-t}$, and $y_2=e^{-2t}$

Differential equations - free response

- Free response – what output would be if input was zero
  - i.e. have differential equation: $\sum_{i=0}^{N} a_i \frac{d^i y}{dt^i} = 0$
  - Solution $y(t)$ depends only on initial conditions: $\frac{d^i}{dt^i}y(0), \quad i=0...N$
- Can express the free response solution as a weighted sum of independent functions: $y(t)=\sum_{i=1}^{n} c_i y_i(t)$
  where constants depend on initial conditions
- Example $\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=x$

  free response is $y_a(t) = c_1e^{-t} + c_2e^{-2t}$

Differential equations
- forced response

- Forced response is solution if initial conditions are zero

  i.e. $\frac{d^i}{dt^i}y(0)=0, \quad i=0...n$

  Forced response depends *only* on the input, and for linear system can be written as a convolution integral:

  $$y_b(t)=\int_0^t w(t-\tau)x(\tau)d\tau$$

  where w(t) is the weighting function of the differential equation and can be written as a sum of the solution set to the free response:

  $$w(t)=\sum_{i=0}^{n}c_i y_i(t) \qquad t\geq 0$$

  with initial conditions:

  $$w(0)=0, \quad \left.\frac{dw}{dt}\right|_{t=0}=0, \quad \left.\frac{d^{n-2}w}{dt^{n-2}}\right|_{t=0}=0, \quad \left.\frac{d^{n-1}w}{dt^{n-1}}\right|_{t=0}=1$$

Differential equations
- forced response

- Example $\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=x$

- From characteristic solution $w(t)=c_1e^{-t}+c_2e^{-2t}$

- Set initial conditions $w(0)=0$, $\left.\frac{dw}{dt}\right|_{t=0}=1$

  $c_1 + c_2 = 0$, $-c_1 + -2c_2 = 1$,

  So $c_1 = 1, c_2 = -1$

- E.g. if $x(t) = 1$, then forced response is

  $$y_b(t)=\int_0^t\left[e^{-(t-\tau)}-e^{-2(t-\tau)}\right]1d\tau$$

  $$=e^{-t}\int_0^t e^{\tau}d\tau - e^{-2t}\int_0^t e^{2\tau}d\tau \qquad =\tfrac{1}{2}\left(1-2e^{-t}+e^{-2t}\right)$$

Transform domain
- Laplace transform

- Recall that solutions to differential equations can be expressed in terms of real and complex exponentials
  - Can manipulate coefficients algebraically
- So, define *Laplace Transform* of a function of time

$$L\{f(t)\} \equiv F(s) \equiv \lim_{\substack{T\to\infty \\ \varepsilon\to 0}} \int_{\varepsilon}^{T} f(t)e^{-st}\,dt$$

  where *s* is a complex variable $s = \alpha + j\omega$
- Note that Laplace transform may not converge for all values of s. In general, will be value σ_0 for which

$$\lim_{\substack{T\to\infty \\ \varepsilon\to 0}} \int_{\varepsilon}^{T} |f(t)|\, e^{-\sigma_0 t}\,dt < +\infty$$

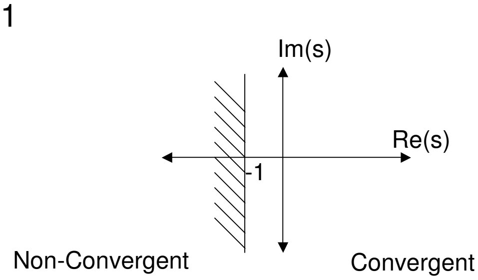
  i.e. Laplace transform exists only for Re(s) > σ_0

Laplace transform

- E.g. Laplace transform of $f(t) = e^{-t}$

$$L\{e^{-t}\} = \int_{0+}^{\infty} e^{-t}e^{-st}\,dt = \frac{-1}{s+1}e^{-(s+1)t}\Big|_{0+}^{\infty} = \frac{1}{s+1}$$

  but only for Re(s) > -1



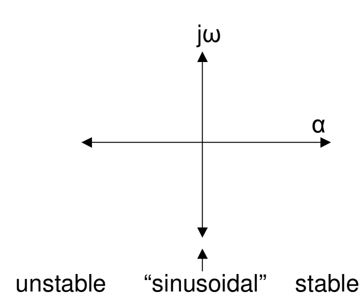
Laplace transform properties

- Derivative: $L\left\{\frac{df}{dt}\right\} = sF(s) - f(0^+)$
- Integral: $L\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s)$
- Convolution: $L\left\{\int_{0+}^t f_1(\tau)f_2(t-\tau)d\tau\right\} = F_1(s).F_2(s)$
- Time-shift: $L\{f(t-T)\} = e^{-sT}F(s)$
- Time-scaling: $L\{f(t/a)\} = aF(as)$

Laplace domain

- Laplace variable s = α + jω
- i.e. α is damping term $e^{-\alpha t}$

  ω is oscillatory term $e^{-j\omega t}$
- So, solution on Laplace "plane" gives representation of oscillatory and damping behaviour of signal / system
  - Note instability if α<0 (ie, LHP).



Laplace example

- 2nd order system $a_1 \frac{d^2 y}{dt^2} + a_2 \frac{dy}{dt} + a_3 y = bx$
- Laplace transform (zero initial conditions)

$$a_1 s^2 Y + a_2 sY + a_3 Y = bX$$

$$Y(s) = \frac{X(s)}{a_1 s^2 + a_2 s + a_3}$$

i.e. Algebraic relationship between input and output – *transfer function*

$$G(s) = \frac{Y(s)}{X(s)}$$

Summary

- Can represent (physical) system by mathematical system of signals and operations
- Express system in terms of differential equations
  - Solve to obtain behaviour of system
  - Weighted sums of fundamental solution sets
- Transform time domain to (complex) frequency domain
  - Laplace transform
  - Differential and integral operations become algebraic operations
- In transform domain, system transfer function becomes an algebraic expression between input and output

- Goodwin, Graebe, Salgado: Chapter 4 (modelling), chapter 3 (state-space)